**4-8** 

# **Using Matrices to Solve Systems of Equations**

#### **Main Ideas**

- Write matrix equations for systems of equations.
- Solve systems of equations using matrix equations.

#### **New Vocabulary**

matrix equation

#### GET READY for the Lesson

An ecologist is studying two species of birds that compete for food and territory. He estimates that a particular region with an area of 14.25 acres (approximately 69,000 square yards) can supply 20,000 pounds of food for the birds.

Species A needs 140 pounds of food and has a territory of 500 square yards per nesting pair. Species B needs 120 pounds of food and has a territory of 400 square yards per nesting pair. The biologist can use this information to find the number of birds of each species that the area can support.



**Write Matrix Equations** The situation above can be represented using a system of equations that can be solved using matrices. Let's examine a similar situation. Consider the system of equations below. You can write this system with matrices by using the left and right sides of the equations.

$$5x + 7y = 11$$
  

$$3x + 8y = 18 \rightarrow \begin{bmatrix} 5x + 7y \\ 3x + 8y \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \end{bmatrix}$$

Write the matrix on the left as the product of the coefficient matrix and the variable matrix.

Α	•	X	=	В
$\begin{bmatrix} 5 & 7 \\ 3 & 8 \end{bmatrix}$	•	$\begin{bmatrix} x \\ y \end{bmatrix}$	=	$\begin{bmatrix} 11\\18 \end{bmatrix}$
coefficient matrix		variable matrix		constant matrix

The system of equations is now expressed as a **matrix equation**.

### EXAMPLE Two-Variable Matrix Equation

Write a matrix equation for the system of equations.

$$5x - 6y = -47$$
$$3x + 2y = -17$$

Determine the coefficient, variable, and constant matrices.

 $\begin{array}{ccc} 5x - 6y = -47 \\ 3x + 2y = -17 \end{array} \rightarrow \begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} -47 \\ -17 \end{bmatrix}$ 

Write the matrix equation.  

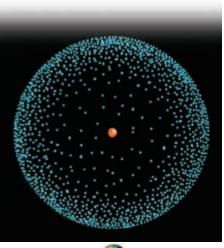
$$A \cdot X = B$$

$$\begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -47 \\ -17 \end{bmatrix}$$

$$A \in CK$$

$$A = CK$$

$$A$$



Real-World Link ....

Atomic mass units (amu) are relative units of weight because they were compared to the weight of a hydrogen atom. So a molecule of nitrogen, whose weight is 14.0 amu, weighs 14 times as much as a hydrogen atom.

Source: www.sizes.com

### Real-World EXAMPLE

**CHEMISTRY** The molecular formula for glucose is  $C_6H_{12}O_6$ , which represents that a molecule of glucose has 6 carbon (C) atoms, 12 hydrogen (H) atoms, and 6 oxygen (O) atoms. One molecule of glucose weighs 180 atomic mass units (amu), and one oxygen atom weighs 16 amu. The formulas and weights for glucose and sucrose are listed below.

Sugar	Formula	Atomic Weight (amu)
glucose	C <sub>6</sub> H <sub>12</sub> O <sub>6</sub>	180
sucrose	C <sub>12</sub> H <sub>22</sub> O <sub>11</sub>	342

**a**. Write a system of equations that represents the weight of each atom. Let *c* represent the weight of a carbon atom.

Let *h* represent the weight of a hydrogen atom.

Glucose:	6c + 12h + 6(16) = 180	Equation for glucose
	6c + 12h + 96 = 180	Simplify.
	6c + 12h = 84	Subtract 96 from each side.
Sucrose:	12c + 22h + 11(16) = 342	Equation for sucrose
	12c + 22h + 176 = 342	Simplify.
	12c + 22h = 166	Subtract 176 from each side.

#### **b.** Write a matrix equation for the system of equations.

Determine the coefficient, variable, and constant matrices. Then write the matrix equation.

$$\begin{array}{rcl} 6c+12h=84\\ 12c+22h=166 \end{array} \rightarrow \begin{bmatrix} 6 & 12\\ 12 & 22 \end{bmatrix} \cdot \begin{bmatrix} c\\ h \end{bmatrix} = \begin{bmatrix} 84\\ 166 \end{bmatrix}$$
$$\begin{array}{rcl} A & \cdot & X & = & B\\ \begin{bmatrix} 6 & 12\\ 12 & 22 \end{bmatrix} \cdot \begin{bmatrix} c\\ h \end{bmatrix} = \begin{bmatrix} 84\\ 166 \end{bmatrix}$$
You will solve this matrix equation in Exercise 3.

### CHECK Your Progress

**2.** The formula for propane is  $C_3H_8$ , and its atomic weight is 44 amu. Butane is  $C_4H_{10}$ , and its atomic weight is 58 amu. Write a system of equations for the weight of each. Then write a matrix equation for the system of equations.



### **Study Tip**

#### Solving Using Inverses

Notice that  $A^{-1}$  is on the left on both sides of the equation. It is important to multiply both sides of the matrix equation with the inverse in the same order since matrix multiplication is not commutative. **Solve Systems of Equations** A matrix equation in the form AX = B, where A is a coefficient matrix, X is a variable matrix, and B is a constant matrix, can be solved in a similar manner as a linear equation of the form ax = b.

ax = b	Write the equation.	AX = B
$\left(\frac{1}{a}\right)ax = \left(\frac{1}{a}\right)b$	Multiply each side by the inverse of the coefficient, if it exists.	$A^{-1}AX = A^{-1}B$
$1x = \left(\frac{1}{a}\right)b$	$\left(\frac{1}{a}\right)a = 1, A^{-1}A = I$	$IX = A^{-1}B$
$x = \left(\frac{1}{a}\right)b$	1x = x, IX = X	$X = A^{-1}B$

Notice that the solution of the matrix equation is the product of the inverse of the coefficient matrix and the constant matrix.

### EXAMPLE Solve Systems of Equations

🚺 Use a matrix equation to solve each system of equations.

**a.** 6x + 2y = 11 3x - 8y = 1The matrix equation is  $\begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$ , when  $A = \begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix}$ ,  $X = \begin{bmatrix} x \\ y \end{bmatrix}$ , and  $B = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$ .

**Step 1** Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{-48 - 6} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \text{ or } -\frac{1}{54} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix}$$

**Step 2** Multiply each side of the matrix equation by the inverse matrix.

 $-\frac{1}{54} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 & -2 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 11 \\ 1 \end{bmatrix}$  Multiply each side by  $A^{-1}$ .  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54} \begin{bmatrix} -90 \\ -27 \end{bmatrix}$  Multiply matrices.  $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{2} \end{bmatrix}$   $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ The solution is  $\left(\frac{5}{3}, \frac{1}{2}\right)$ . Check this solution in the original equation.

**b.** 6a - 9b = -18 8a - 12b = 24The matrix equation is  $\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$ , when  $A = \begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}$ ,  $X = \begin{bmatrix} a \\ b \end{bmatrix}$ , and  $B = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$ .

# Study Tip

Identity Matrix The identity matrix on the left verifies that the inverse matrix has been calculated correctly. Find the inverse of the coefficient matrix.

 $A^{-1} = \frac{1}{-72+72} \begin{bmatrix} -12 & 9\\ -8 & 6 \end{bmatrix}$ 

The determinant of the coefficient matrix

$$\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}$$
 is 0, so  $A^{-1}$  does not exist.

There is no unique solution of this system.

Graph the system of equations. Since the lines are parallel, this system has no solution. Therefore, the system is inconsistent.

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CHECK Your Progress **3A.** -2x + 3y = -74x - 8y = 16

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**3B.** 2x - 4y = -243x - 6y = -12

To solve a system of equations with three variables, you can use the  $3 \times 3$ identity matrix. However, finding the inverse of a  $3 \times 3$  matrix may be tedious. Graphing calculators and computers offer fast and accurate calculations.

### **GRAPHING CALCULATOR LAB**

#### Systems of Three Equations in Three Variables

You can use a graphing calculator and a matrix equation to solve systems of equations. Consider the system of equations below.

3x-2y+z=02x + 3y - z = 175x - y + 4z = -7

#### THINK AND DISCUSS

- 1. Write a matrix equation for the system of equations.
- **2.** Enter the coefficient matrix as matrix A and the constant matrix as matrix B. Find the product of  $A^{-1}$  and B. Recall that the  $x^{-1}$  key is used to find  $A^{-1}$ .
- **3.** How is the result related to the solution?

## Your Understanding

Example 1	Write a matrix equation for each system of equations.		
(pp. 216–217)	<b>1.</b> $x - y = -3$	<b>2.</b> $2g + 3h = 8$	
	x + 3y = 5	-4g - 7h = -5	

Example 2 **3. CHEMISTRY** Refer to Example 2 on page 217. Solve the system of equations (p. 217) to find the weight of a carbon, hydrogen, and oxygen atom.

#### Lesson 4-8 Using Matrices to Solve Systems of Equations 219

#### **Review** Vocabulary

**Inconsistent System** of Equations: a system of equations that does not have a solution (Lesson 3-1)

(pp. 218-219)

**4.** 5x - 3y = -30**5.** 5s + 4t = 128x + 5y = 14s - 3t = -1.25**6.** 3x + 6y = 11**7.** 3x + 4y = 36x + 8y = 52x + 4y = 7

### Exercises

HOMEWORK HELP			
For Exercises	See Examples		
8–11	1		
12, 13	2		
14–23	3, 4		

#### Write a matrix equation for each system of equations.

<b>8.</b> $3x - y = 0$	<b>9.</b> $4x - 7y = 2$
x + 2y = -21	3x + 5y = 9
<b>10.</b> $5a - 6b = -47$	<b>11.</b> $3m - 7n = -43$
3a + 2b = -17	6m + 5n = -10

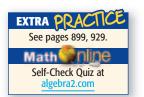
- **12.** MONEY Mykia had 25 quarters and dimes. The total value of all the coins was \$4.00. How many quarters and dimes did Mykia have?
- **13. PILOT TRAINING** Flight instruction costs \$105 per hour, and the simulator costs \$45 per hour. Hai-Ling spent 4 more hours in airplane training than in the simulator. If Hai-Ling spent \$3870, how much time did he spend training in an airplane and in a simulator?

#### Use a matrix equation to solve each system of equations.

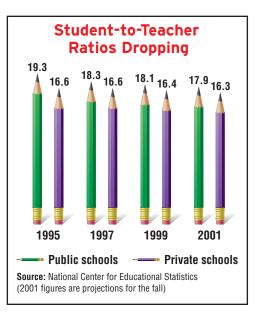
<b>14.</b> $p - 2q = 1$	<b>15.</b> $3x - 9y = 12$
p + 5q = 22	-2x + 6y = 9
<b>16.</b> $-2x + 4y = 3$	<b>17.</b> $6r + s = 9$
2x - 4y = 5	3r = -2s
<b>18.</b> $5a + 9b = -28$	<b>19.</b> $6x - 10y = 7$
2a - b = -2	3x - 5y = 8
<b>20.</b> $4m - 7n = -63$	<b>21.</b> $8x - 3y = 19.5$
3m + 2n = 18	2.5x + 7y = 18
<b>22.</b> $x + 2y = 8$	<b>23.</b> $4x - 3y = 5$
3x + 2y = 6	2x + 9y = 6

- **24.** NUMBER THEORY Find two numbers whose sum is 75 and the second number is 15 less than twice the first.
- 25. CHEMISTRY Refer to Check Your Progress 2 on page 217. Solve the system of equations to find the weights of a carbon and a hydrogen atom.
- **26. SPORTING GOODS** Use three rows from the table of sporting goods sales and write a matrix. Then use the matrix to find the cost of each type of ball.

Day	Baseballs	Basketballs	Footballs	Sales (\$)
Monday	10	3	6	97
Tuesday	13	1	4	83
Wednesday	8	5	2	79
Thursday	15	2	7	116
Friday	9	0	8	84



- **27. SCHOOLS** The graphic shows that student-to-teacher ratios are dropping in both public and private schools. If these rates of change remain constant, predict when the student-to-teacher ratios for private and public schools will be the same.
- **28. CHEMISTRY** Cara is preparing an acid solution. She needs 200 milliliters of 48% concentration solution. Cara has 60% and 40% concentration solutions in her lab. How many milliliters of 40% acid solution should be mixed with 60% acid solution to make the required amount of 48% acid solution?



Graphing Calculator

Use a graphing calculator to solve each system of equations using inverse matrices.

- **29.** 2a b + 4c = 6<br/>a + 5b 2c = -6<br/>3a 2b + 6c = 8**30.** 3x 5y + 2z = 22<br/>2x + 3y z = -9<br/>4x + 3y + 3z = 1**31.** 2q + r + s = 2<br/>-q r + 2s = 7<br/>-3q + 2r + 3s = 7
- - **33. OPEN ENDED** Write a system of equations that does not have a unique solution.
  - 34. FIND THE ERROR Tommy and Laura are solving a system of equations.
    - They find that  $A^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$ ,  $B = \begin{bmatrix} -7 \\ -9 \end{bmatrix}$ , and  $X = \begin{bmatrix} x \\ y \end{bmatrix}$ . Who is correct?

Explain your reasoning.

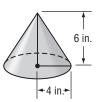
Tommy	Laura
$ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ -9 \end{bmatrix} $	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$
$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 42 \\ 31 \end{bmatrix}$

- **35. CHALLENGE** What can you conclude about the solution set of a system of equations if the coefficient matrix does not have an inverse?
- **36.** *Writing in Math* Use the information about ecology found on page 216 to explain how matrices can be used to find the number of species of birds that an area can support. Demonstrate a system of equations that can be used to find the number of each species the region can support, and a solution of the problem using matrices.

#### STANDARDIZED TEST PRACTICE

- **37. ACT/SAT** The Yogurt Shoppe sells cones in three sizes: small, \$0.89; medium, \$1.19; and large, \$1.39. One day Scott sold 52 cones. He sold seven more medium cones than small cones. If he sold \$58.98 in cones, how many medium cones did he sell?
  - A 11 C 24
  - **B** 17 **D** 36
- **38. ACT/SAT** What is the solution to the system of equations 6a + 8b = 5 and 10a 12b = 2?
  - $F \left(\frac{3}{4}, \frac{1}{2}\right) \qquad H \left(\frac{1}{2}, \frac{3}{4}\right)$  $G \left(\frac{1}{2}, -\frac{1}{2}\right) \qquad J \left(\frac{1}{2}, \frac{1}{4}\right)$

**39. REVIEW** A right circular cone has radius 4 inches and height 6 inches.



What is the lateral area of the cone? (Lateral area of cone =  $\pi r \ell$ , where  $\ell$  = slant height)?

- A  $24\pi$  sq in.
- **B**  $2\sqrt{13\pi}$  sq in.
- C  $2\sqrt{52\pi}$  sq in.
- **D**  $8\sqrt{13\pi}$  sq in.

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5 4

# Spiral Review

Find the inverse of each matrix, if it exists. (Lesson 4-7)

40.	[4	4]	<b>41.</b>	)
40.	2	3]	<b>.</b>	7

Use Cramer's Rule to solve each system of equations. (Lesson 4-6)

**43.** 6x + 7y = 10**44.** 6a + 7b = -10.153x - 4y = 209.2a - 6b = 69.944

**42.**  $\begin{bmatrix} -3 & -6 \\ 5 & 10 \end{bmatrix}$ 

**45.**  $\frac{x}{2} - \frac{2y}{3} = 2\frac{1}{3}$ 3x + 4y = -50

**46. ECOLOGY** If you recycle a  $3\frac{1}{2}$ -foot stack of newspapers, one less 20-foot

loblolly pine tree will be needed for paper. Use a prediction equation to determine how many feet of loblolly pine trees will *not* be needed for paper if you recycle a pile of newspapers 20 feet tall. (Lesson 2-5)

### **Cross-Curricular Project**

#### **Algebra and Consumer Science**

**What Does it Take to Buy a House?** It is time to complete your project. Use the information and data you have gathered about home buying and selling to prepare a portfolio or Web page. Be sure to include your tables, graphs, and calculations in the presentation. You may also wish to include additional data, information, or pictures.

